OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MATHEMATICS 4727

Further Pure Mathematics 3

Thursday **15 JUNE 2006** Afternoon 1 hour 30 minutes

Additional materials:
8 page answer booklet
Graph paper
List of Formulae (MF1)

TIME 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer **all** the questions.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question or is clearly appropriate.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is 72.
- Questions carrying smaller numbers of marks are printed earlier in the paper, and questions carrying larger numbers of marks later in the paper.
- **You are reminded of the need for clear presentation in your answers.**

This question paper consists of 3 printed pages and 1 blank page.

[Turn over

1 **(a)** For the infinite group of non-zero complex numbers under multiplication, state the identity element and the inverse of $1 + 2\text{i}$, giving your answers in the form $a + \text{i}b$. [3]

(b) For the group of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ under matrix addition, where $a \in \mathbb{R}$, state the identity element and the inverse of $\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$. [2]

2 **(a)** Given that $z_1 = 2\text{e}^{\frac{1}{6}\pi\text{i}}$ and $z_2 = 3\text{e}^{\frac{1}{4}\pi\text{i}}$, express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $r\text{e}^{\text{i}\theta}$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [4]

(b) Given that $w = 2(\cos\frac{1}{8}\pi + \text{i}\sin\frac{1}{8}\pi)$, express w^{-5} in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r > 0$ and $0 \leqslant \theta < 2\pi$. [3]

3 Find the perpendicular distance from the point with position vector $12\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ to the line with equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + t(8\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$. [6]

4 Find the solution of the differential equation

$$\frac{\text{d}y}{\text{d}x} - \frac{x^2 y}{1 + x^3} = x^2$$

for which $y = 1$ when $x = 0$, expressing your answer in the form $y = \text{f}(x)$. [8]

5 A line l_1 has equation $\frac{x}{2} = \frac{y+4}{3} = \frac{z+9}{5}$.

(i) Find the cartesian equation of the plane which is parallel to l_1 and which contains the points $(2, 1, 5)$ and $(0, -1, 5)$. [5]

(ii) Write down the position vector of a point on l_1 with parameter t. [1]

(iii) Hence, or otherwise, find an equation of the line l_2 which intersects l_1 at right angles and which passes through the point $(-5, 3, 4)$. Give your answer in the form $\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$. [4]

6 **(i)** Find the general solution of the differential equation

$$\frac{\text{d}^2 y}{\text{d}x^2} + 4y = \sin x.$$ [6]

(ii) Find the solution of the differential equation for which $y = 0$ and $\frac{\text{d}y}{\text{d}x} = \frac{4}{3}$ when $x = 0$. [4]

7 The series C and S are defined for $0 < \theta < \pi$ by

$$C = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta,$$
$$S = \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta.$$

(i) Show that $C + \mathrm{i}S = \dfrac{\mathrm{e}^{3\mathrm{i}\theta} - \mathrm{e}^{-3\mathrm{i}\theta}}{\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta} - \mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta}}\mathrm{e}^{\frac{5}{2}\mathrm{i}\theta}$. [4]

(ii) Deduce that $C = \sin 3\theta \cos \frac{5}{2}\theta \operatorname{cosec} \frac{1}{2}\theta$ and write down the corresponding expression for S. [4]

(iii) Hence find the values of θ, in the range $0 < \theta < \pi$, for which $C = S$. [4]

8 A group D of order 10 is generated by the elements a and r, with the properties $a^2 = e$, $r^5 = e$ and $r^4a = ar$, where e is the identity. Part of the operation table is shown below.

	e	a	r	r^2	r^3	r^4	ar	ar^2	ar^3	ar^4
e	e	a	r	r^2	r^3	r^4	ar	ar^2	ar^3	ar^4
a	a	e	ar	ar^2	ar^3	ar^4				
r	r		r^2	r^3	r^4	e				
r^2	r^2		r^3	r^4	e	r				
r^3	r^3		r^4	e	r	r^2				
r^4	r^4	ar	e	r	r^2	r^3				
ar	ar		ar^2	ar^3	ar^4	a				
ar^2	ar^2		ar^3	ar^4	a	ar		E		
ar^3	ar^3		ar^4	a	ar	ar^2				
ar^4	ar^4		a	ar	ar^2	ar^3				

(i) Give a reason why D is not commutative. [1]

(ii) Write down the orders of any possible proper subgroups of D. [2]

(iii) List the elements of a proper subgroup which contains

(a) the element a, [1]

(b) the element r. [1]

(iv) Determine the order of each of the elements r^3, ar and ar^2. [4]

(v) Copy and complete the section of the table marked **E**, showing the products of the elements ar, ar^2, ar^3 and ar^4. [5]

1 **(a)** Identity = $1+0\,\mathrm{i}$	B1	For correct identity. Allow 1
Inverse = $\dfrac{1}{1+2\mathrm{i}}$	B1	For $\dfrac{1}{1+2\mathrm{i}}$ seen or implied
$= \dfrac{1}{1+2\mathrm{i}} \times \dfrac{1-2\mathrm{i}}{1-2\mathrm{i}} = \frac{1}{5} - \frac{2}{5}\mathrm{i}$	B1 **3**	For correct inverse **AEFcartesian**
(b) Identity = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	B1	For correct identity
Inverse = $\begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$	B1 **2**	For correct inverse
	$\boxed{5}$	
2 **(a)** $(z_1 z_2 =)\ 6\mathrm{e}^{\frac{5}{12}\pi\mathrm{i}}$	B1 B1	For modulus = 6 For argument = $\frac{5}{12}\pi$
$\left(\dfrac{z_1}{z_2} = \frac{2}{3}\mathrm{e}^{-\frac{1}{12}\pi\mathrm{i}} =\right) \frac{2}{3}\mathrm{e}^{\frac{23}{12}\pi\mathrm{i}}$	M1 A1 **4**	For subtracting arguments For correct answer
(b) $\left(w^{-5} =\right) 2^{-5}\,\mathrm{cis}\left(-\frac{5}{8}\pi\right)$	M1 A1	For use of de Moivre For $-\frac{5}{8}\pi$ seen or implied
$= \frac{1}{32}\left(\cos\frac{11}{8}\pi + \mathrm{i}\sin\frac{11}{8}\pi\right)$	A1 **3**	For correct answer (allow 2^{-5} and $\mathrm{cis}\frac{11}{8}\pi$)
	$\boxed{7}$	

3 *EITHER* $\mathbf{c}-\mathbf{a} = \pm[11, 3, -2]$	B1	For vector joining lines
$(\mathbf{c}-\mathbf{a}) \times [8, 3, -6]$	M1*	For attempt at vector product of $\mathbf{c}-\mathbf{a}$ and $[8, 3, -6]$
$\mathbf{n} = \pm[-12, 50, 9]$	A1 √	For obtaining **n**. f.t. from incorrect $\mathbf{c}-\mathbf{a}$
$d = \dfrac{\lvert\mathbf{n}\rvert}{\lvert[8, 3, -6]\rvert}$	M1 (dep*)	For dividing $\lvert\mathbf{n}\rvert$ by magnitude of $[8, 3, -6]$
$= \dfrac{\sqrt{2725}}{\sqrt{109}}$	A1	For either magnitude correct
$(d =)\ 5$	A1	For correct distance **CAO**
OR $\mathbf{c}-\mathbf{a} = \pm[11, 3, -2]$	B1	For vector joining lines
$(\mathbf{c}-\mathbf{a}) \,.\, [8, 3, -6]$	M1*	For attempt at scalar product of $\mathbf{c}-\mathbf{a}$ and $[8, 3, -6]$
$\cos\theta = \pm\dfrac{109}{\sqrt{134}\sqrt{109}} = \pm\sqrt{\dfrac{109}{134}}$	A1 √	For correct $\cos\theta$ **AEF**. f.t. from incorrect $\mathbf{c}-\mathbf{a}$
$d = \sqrt{134}\sin\theta$	M1 (dep*)	For using trigonometry for perpendicular distance
	A1	For correct expression for d in terms of θ
$(d =)\ 5$	A1	For correct distance **CAO**
OR $\mathbf{c}-\mathbf{a} = \pm[11, 3, -2]$	B1	For vector joining lines
$(\mathbf{c}-\mathbf{a}) \,.\, [8, 3, -6]$	M1*	For attempt at scalar product of $\mathbf{c}-\mathbf{a}$ and $[8, 3, -6]$
$x = \dfrac{109}{\sqrt{109}} = \sqrt{109}$	A1 √	For finding projection of $\mathbf{c}-\mathbf{a}$ onto $[8, 3, -6]$ f.t. from incorrect $\mathbf{c}-\mathbf{a}$
$d = \sqrt{134-109}$	M1 (dep*)	For using Pythagoras for perpendicular distance
	A1	For correct expression for d
$(d =)\ 5$	A1	For correct distance **CAO**
OR $\mathbf{CP} = \pm[-11+8t, -3+3t, 2-6t]$	B1	For finding a vector from $C(12, 5, 3)$ to a point on the line
$\mathbf{CP} \,.\, [8, 3, -6] = 0$	M1*	For using scalar product for perpendicularity
$t = \pm 1$ *OR* $P = (9, 5, -1)$	A1 √	For correct point. f.t. from incorrect **CP**
$d = \sqrt{3^2 + 0^2 + 4^2}$	M1 (dep*)	For finding magnitude of **CP**
	A1	For correct expression for d
$(d =)\ 5$	A1 **6**	For correct distance **CAO**
		SR Obtain
		$\mathbf{CP} = [11, 3, -2] - [8, 3, -6] = \pm[3, 0, 4]$ B1
		Verify $[3, 0, 4] \,.\, [8, 3, -6] = 0$ M1*
		$d = \sqrt{3^2 + 0^2 + 4^2} = 5$ M1(dep*) A1 A1 (maximum 5 / 6)
	$\boxed{6}$	

4 Integrating factor $\mathrm{e}^{\int -\frac{x^2}{1+x^3}\mathrm{d}x}$	M1	For correct process for finding integrating factor
$= \mathrm{e}^{-\frac{1}{3}\ln(1+x^3)} = \left(1+x^3\right)^{-\frac{1}{3}}$	A1	For correct IF, simplified (here or later)
$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(y\left(1+x^3\right)^{-\frac{1}{3}}\right) = \frac{x^2}{\left(1+x^3\right)^{\frac{1}{3}}}$	M1	For multiplying through by their IF
$\Rightarrow y\left(1+x^3\right)^{-\frac{1}{3}} = \frac{1}{2}\left(1+x^3\right)^{\frac{2}{3}}\ (+c)$	M1	For integrating RHS to obtain $A\left(1+x^3\right)^k$ *OR* $\ln A\left(1+x^3\right)^k$
	A1	For correct integration (+*c* not required here)
$\Rightarrow 1 = \frac{1}{2} + c \quad \Rightarrow \quad c = \frac{1}{2}$	M1	For substituting (0, 1) into GS (including + *c*)
	A1 √	For correct *c*. f.t. from their GS
$\Rightarrow y = \frac{1}{2}\left(1+x^3\right) + \frac{1}{2}\left(1+x^3\right)^{\frac{1}{3}}$	A1	For correct solution. **AEF** in form *y* = f(*x*)
	$\boxed{8}$	
5 **(i)** *EITHER* $\mathbf{a} = [2, 3, 5]$, $\mathbf{b} = \pm[2, 2, 0]$	B1	For stating 2 vectors in the plane
$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \pm k[-10, 10, -2]$	M1	For finding perpendicular to plane
	A1 √	For correct **n**. f.t. from incorrect **b**
Use (2, 1, 5) *OR* (0, –1, 5)	M1	For substituting a point into equation $ax + by + cz = d$ where $[a, b, c]$ = their **n**
$\Rightarrow 5x - 5y + z = 10$	A1	For correct cartesian equation **AEF**
OR $\mathbf{a} = [2, 3, 5]$, $\mathbf{b} = \pm[2, 2, 0]$	B1	For stating 2 vectors in the plane
e.g. $\mathbf{r} = [2, 1, 5] + \lambda[2, 2, 0] + \mu[2, 3, 5]$	M1	For stating parametric equation of plane
$[x, y, z] = [2 + 2\lambda + 2\mu, 1 + 2\lambda + 3\mu, 5 + 5\mu]$	A1 √	For writing 3 equations in *x*, *y*, *z* f.t. from incorrect **b**
	M1	For eliminating λ and μ
$\Rightarrow 5x - 5y + z = 10$	A1 **5**	For correct cartesian equation **AEF**
(ii) $[2t, 3t - 4, 5t - 9]$	B1 **1**	For stating a point *A* on l_1 with parameter *t* **AEF**
(iii) $\pm[2t + 5, 3t - 7, 5t - 13]$	M1	For finding direction of l_2 from *A* and (–5,3, 4)
$\pm[2t + 5, 3t - 7, 5t - 13] \,.\, [2, 3, 5] = 0$	M1	For using scalar product for perpendicularity with any vector involving *t*
$\Rightarrow t = 2$	A1	For correct value of *t*
$\frac{x+5}{9} = \frac{y-3}{-1} = \frac{z-4}{-3}$ *OR* $\frac{x-4}{9} = \frac{y-2}{-1} = \frac{z-1}{-3}$	A1 **4**	For a correct equation **AEFcartesian**
		SR For $2p + 3q + 5r = 0$ and no further progress award B1
	$\boxed{10}$	

6 **(i)** $\left(m^2+4=0\Rightarrow\right) m=\pm 2\mathrm{i}$	B1	For correct solutions of auxiliary equation (may be implied by correct CF)
CF = $A\cos 2x+B\sin 2x$	B1	For correct CF (**AEtrig** but not $A\mathrm{e}^{2\mathrm{i}x}+B\mathrm{e}^{-2\mathrm{i}x}$ only)
PI = $p\sin x\ (+\,q\cos x)$	B1	State a trial PI with at least $p\sin x$
$-p\sin x\ (-q\cos x)+4p\sin x\ (+4q\cos x)=\sin x$	M1	For substituting PI into DE
$\Rightarrow p=\frac{1}{3},\quad q=0$	A1	For correct p and q (which may be implied)
$\Rightarrow y=A\cos 2x+B\sin 2x+\frac{1}{3}\sin x$	B1 √ **6**	For using GS = CF + PI, with 2 arbitrary constants in CF and none in PI
(ii) $(0,0)\Rightarrow A=0$	B1 √	For correct equation in A and/or B f.t. from their GS
$\frac{\mathrm{d}y}{\mathrm{d}x}=2B\cos 2x+\frac{1}{3}\cos x\Rightarrow\frac{4}{3}=2B+\frac{1}{3}$	M1	For differentiating their GS and substituting values for x and $\frac{\mathrm{d}y}{\mathrm{d}x}$
$A=0,\ B=\frac{1}{2}$	A1	For correct A and B Allow $A=-\frac{1}{4}\mathrm{i},\ B=\frac{1}{4}\mathrm{i}$ from CF $A\mathrm{e}^{2\mathrm{i}x}+B\mathrm{e}^{-2\mathrm{i}x}$
$\Rightarrow y=\frac{1}{2}\sin 2x+\frac{1}{3}\sin x$	A1 **4**	For stating correct solution **CAO**
	$\boxed{10}$	
7 **(i)** $C+\mathrm{i}S=1+\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{2\mathrm{i}\theta}+\mathrm{e}^{3\mathrm{i}\theta}+\mathrm{e}^{4\mathrm{i}\theta}+\mathrm{e}^{5\mathrm{i}\theta}$	M1	For using de Moivre, showing at least 3 terms
$=\frac{\mathrm{e}^{6\mathrm{i}\theta}-1}{\mathrm{e}^{\mathrm{i}\theta}-1}$	M1 A1	For recognising GP For correct GP sum
$=\frac{\mathrm{e}^{3\mathrm{i}\theta}-\mathrm{e}^{-3\mathrm{i}\theta}}{\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}-\mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta}}\cdot\frac{\mathrm{e}^{3\mathrm{i}\theta}}{\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}}=\frac{\mathrm{e}^{3\mathrm{i}\theta}-\mathrm{e}^{-3\mathrm{i}\theta}}{\mathrm{e}^{\frac{1}{2}\mathrm{i}\theta}-\mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta}}\,\mathrm{e}^{\frac{5}{2}\mathrm{i}\theta}$	A1 **4**	For obtaining correct expression **AG**
(ii) $C+\mathrm{i}S=\frac{2\mathrm{i}\sin 3\theta}{2\mathrm{i}\sin\frac{1}{2}\theta}\ .\ \mathrm{e}^{\frac{5}{2}\mathrm{i}\theta}$	M1 A1	For expressing numerator and denominator in terms of sines For $k\sin 3\theta$ and $k\sin\frac{1}{2}\theta$
$\mathrm{Re}\Rightarrow C=\sin 3\theta\cos\frac{5}{2}\theta\operatorname{cosec}\frac{1}{2}\theta$	A1	For correct expression **AG**
$\mathrm{Im}\Rightarrow S=\sin 3\theta\sin\frac{5}{2}\theta\operatorname{cosec}\frac{1}{2}\theta$	B1 **4**	For correct expression
(iii) $C=S\ \Rightarrow\ \sin 3\theta=0,\ \tan\frac{5}{2}\theta=1$	M1	For either equation deduced **AEF**
		Ignore values outside $0<\theta<\pi$
$\theta=\frac{1}{3}\pi,\frac{2}{3}\pi$	A1	For both values correct and no extras
$\theta=\frac{1}{10}\pi,\frac{1}{2}\pi,\frac{9}{10}\pi$	A2 **4**	For all values correct and no extras. Allow A1 for any 1 value *OR* all correct with extras
	$\boxed{12}$	

8 (i) $r^4.a \neq a.r^4$	B1 **1**	For stating the non-commutative product in the given table, or justifying another correct one
(ii) Possible subgroups order 2, 5	B1 B1 **2**	For either order stated For both orders stated, and no more (Ignore 1)
(iii) (a) $\{e, a\}$	B1	For correct subgroup
(b) $\{e, r, r^2, r^3, r^4\}$	B1 **2**	For correct subgroup
(iv) order of $r^3 = 5$	B1	For correct order
$(ar)^2 = ar.ar = r^4a.ar = e$	M1	For attempt to find $(ar)^m = e$ OR $(ar^2)^m = e$
$\Rightarrow$ order of $ar = 2$	A1	For correct order
$(ar^2)^2 = ar^2ar.r = ar^2r^4a.r = ara.r = e$		
$\Rightarrow$ order of $ar^2 = 2$	A1 **4**	For correct order
(v)		If the border elements ar ar^2 ar^3 ar^4 are not written, it will be assumed that the products arise from that order
$\begin{array}{c\|cccc} & ar & ar^2 & ar^3 & ar^4 \\ \hline ar & e & r & r^2 & r^3 \\ ar^2 & r^4 & e & r & r^2 \\ ar^3 & r^3 & r^4 & e & r \\ ar^4 & r^2 & r^3 & r^4 & e \end{array}$	B1 B1 B1 B1 B1 **5**	For all 16 elements of the form e or r^m For all 4 elements in leading diagonal = e For no repeated elements in any completed row or column For any two rows or columns correct For all elements correct
	$\boxed{14}$	